3.E: Multiple Integrals (Exercises)

3.1: Double Integrals

A

For Exercises 1-4, find the volume under the surface $z = f(x, y)$ over the rectangle R.

3.1.1. $f(x, y) = 4xy,\ R = [0, 1] \times [0, 1]$

3.1.2. $f(x, y) = e^{x+y},\ R = [0, 1] \times [-1, 1]$

3.1.3. $f(x, y) = x^3 + y^2,\ R = [0, 1] \times [0, 1]$

3.1.4. $f(x, y) = x^4 + xy + y^3,\ R = [1, 2] \times [0, 2]$

For Exercises 5-12, evaluate the given double integral.

3.1.5. $\int_0^1 \int_1^2 (1 - y)x^2\, dx\, dy$

3.1.6. $\int_0^1 \int_0^2 x(x + y)\, dx\, dy$

3.1.7. $\int_0^2 \int_0^1 (x + 2)\, dx\, dy$

3.1.8. $\int_{-1}^2 \int_{-1}^1 x(xy + \sin x)\, dx\, dy$

3.1.9. $\int_0^{\pi/2} \int_0^1 xy \cos(x^2 y)\, dx\, dy$

3.1.10. $\int_0^{\pi} \int_0^{\pi/2} \sin x \cos(y - \pi)\, dx\, dy$

3.1.11. $\int_0^2 \int_1^4 xy\, dx\, dy$

3.1.12. $\int_{-1}^1 \int_{-1}^2 1\, dx\, dy$

3.1.13. Let M be a constant. Show that $\int_c^d \int_a^b M\, dx\, dy = M(d - c)(b - a)$.

3.2: Double Integrals Over a General Region

A

For Exercises 1-6, evaluate the given double integral.

3.2.1. $\int_0^1 \int_{\sqrt{x}}^1 24x^2 y\, dy\, dx$

3.2.2. $\int_0^{\pi} \int_0^y \sin x\, dx\, dy$

3.2.3. $\int_1^2 \int_0^{\ln x} 4x\, dy\, dx$

3.2.4. $\int_0^2 \int_0^{2y} e^{y^2}\, dx\, dy$

3.2.5. $\int_0^{\pi/2} \int_0^y \cos x \sin y\, dx\, dy$

3.2.6. $\int_0^{\infty} \int_0^{\infty} xye^{-(x^2+y^2)}\, dx\, dy$

3.2.7. $\int_0^2 \int_0^y 1\, dx\, dy$

3.2.8. $\int_0^1 \int_0^{x^2} 2\, dy\, dx$

3.2.9. Find the volume V of the solid bounded by the three coordinate planes and the plane $x + y + z = 1$.

3.2.10. Find the volume V of the solid bounded by the three coordinate planes and the plane $3x + 2y + 5z = 6$.

B

3.2.11. Explain why the double integral $\iint\limits_R 1\, dA$ gives the area of the region R. For simplicity, you can assume that R is a region of the type shown in Figure 3.2.1(a).

C

3.2.12. Prove that the volume of a tetrahedron with mutually perpendicular adjacent sides of lengths a, b, and c, as in Figure 3.2.6, is $\frac{abc}{6}$. (*Hint: Mimic Example 3.5, and recall from Section 1.5 how three noncollinear points determine a plane.*)

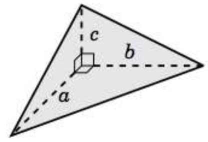

Figure 3.2.6

3.2.13. Show how Exercise 12 can be used to solve Exercise 10.

3.3: Triple Integrals

A

For Exercises 1-8, evaluate the given triple integral.

3.3.1. $\int_0^3\int_0^2\int_0^1 xyz\,dx\,dy\,dz$

3.3.2. $\int_0^1\int_0^x\int_0^y xyz\,dz\,dy\,dx$

3.3.3. $\int_0^\pi\int_0^x\int_0^{xy} x^2\sin z\,dz\,dy\,dx$

3.3.4. $\int_0^1\int_0^z\int_0^y ze^{y^2}\,dx\,dy\,dz$

3.3.5. $\int_1^e\int_0^y\int_0^{1/y} x^2z\,dx\,dz\,dy$

3.3.6. $\int_1^2\int_0^{y^2}\int_0^{z^2} yz\,dx\,dz\,dy$

3.3.7. $\int_1^2\int_2^4\int_0^3 1\,dx\,dy\,dz$

3.3.8. $\int_0^1\int_0^{1-x}\int_0^{1-x-y} 1\,dz\,dy\,dx$

3.3.9. Let M be a constant. Show that $\int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} M\,dx\,dy\,dz = M(z_2-z_1)(y_2-y_1)(x_2-x_1)$.

B

3.3.10. Find the volume V of the solid S bounded by the three coordinate planes, bounded above by the plane $x+y+z=2$, and bounded below by the plane $z=x+y$.

C

3.3.11. Show that $\int_a^b\int_a^z\int_a^y f(x)\,dx\,dy\,dz = \int_a^b \frac{(b-x)^2}{2}f(x)\,dx$. (*Hint: Think of how changing the order of integration in the triple integral changes the limits of integration.*)

3.4: Numerical Approximation of Multiple Integrals

C

3.4.1. Write a program that uses the Monte Carlo method to approximate the double integral $\iint\limits_R e^{xy}\,dA$, where $R=[0,1]\times[0,1]$. Show the program output for $N = 10,\ 100,\ 1000,\ 10000,\ 100000$ and 1000000 random points.

3.4.2. Write a program that uses the Monte Carlo method to approximate the triple integral \iiint\limits_S e^{ x yz}\, dV\), where $S=[0,1]\times[0,1]\times[0,1]$. Show the program output for $N = 10,\ 100,\ 1000,\ 10000,\ 100000$ and 1000000 random points.

3.4.3. Repeat Exercise 1 with the region $R = (x,y) : -1\le x\le 1,\ 0\le y\le x^2$.

3.4.4. Repeat Exercise 2 with the solid $S = (x,y,z) : 0\le x\le 1,\ 0\le y\le 1,\ 0\le z\le 1-x-y$.

3.4.5. Use the Monte Carlo method to approximate the volume of a sphere of radius 1.

3.4.6. Use the Monte Carlo method to approximate the volume of the ellipsoid $\frac{x^2}{9}+\frac{y^2}{4}+\frac{z^2}{1}=1$.

3.5: Change of Variables in Multiple Integrals

A

3.5.1. Find the volume V inside the paraboloid $z=x^2+y^2$ for $0\le z\le 4$.

3.5.2. Find the volume V inside the cone $z=\sqrt{x^2+y^2}$ for $0\le z\le 3$.

B

3.5.3. Find the volume V of the solid inside both $x^2+y^2+z^2=4$ and $x^2+y^2=1$.

3.5.4. Find the volume V inside both the sphere $x^2+y^2+z^2=1$ and the cone $z=\sqrt{x^2+y^2}$.

3.5.5. Prove Equation (3.25).

3.5.6. Prove Equation (3.26).

3.5.7. Evaluate $\iint\limits_R \sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) dA$, where R is the triangle with vertices $(0,0)$, $(2,0)$ and $(1,1)$. (*Hint: Use the change of variables* $u=(x+y)/2,\ v=(x-y)/2.$)

3.5.8. Find the volume of the solid bounded by $z=x^2+y^2$ and $z^2=4(x^2+y^2)$.

3.5.9. Find the volume inside the elliptic cylinder $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ for $0\le z\le 2$.

C

3.5.10. Show that the volume inside the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is $\frac{4\pi abc}{3}$. (*Hint: Use the change of variables* $x=au,\ y=bv,\ z=cw$ *, then consider Example 3.12.*)

3.5.11. Show that the *Beta function*, defined by

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt,\ \text{for } x>0,\ y>0, \tag{3.E.1}$$

satisfies the relation $B(y,x)=B(x,y)$ for $x>0,\ y>0$.

3.5.12. Using the substitution $t=u/(u+1)$, show that the Beta function can be written as

$$B(x,y)=\int_0^\infty \frac{u^{x-1}}{(u+1)^{x+y}}du,\ \text{for } x>0,\ y>0. \tag{3.E.2}$$

3.6: Application: Center of Mass

A

For Exercises 1-5, find the center of mass of the region R with the given density function $\delta(x,y)$.

3.6.1. $R=(x,y):0\le x\le 2,\ 0\le y\le 4,\ \delta(x,y)=2y$

3.6.2. $R=(x,y):0\le x\le 1,\ 0\le y\le x2,\ \delta(x,y)=x+y$

3.6.3. $R=(x,y):y\ge 0,\ x^2+y^2\le a^2,\ \delta(x,y)=1$

3.6.4. $R=(x,y):y\ge 0,\ x\ge 0,\ 1\le x^2+y^2\le 4,\ \delta(x,y)=\sqrt{x^2+y^2}$

3.6.5. $R=(x,y):y\ge 0,\ x^2+y^2\le 1,\ \delta(x,y)=y$

B

For Exercises 6-10, find the center of mass of the solid S with the given density function $\delta(x,y,z)$.

3.6.6. $S=(x,y,z):0\le x\le 1,\ 0\le y\le 1,\ 0\le z\le 1,\ \delta(x,y,z)=xyz$

3.6.7. $S=(x,y,z):z\ge 0,\ x^2+y^2+z^2\le a^2,\ \delta(x,y,z)=x^2+y^2+z^2$

3.6.8. $S=(x,y,z):x\ge 0,\ y\ge 0,\ z\ge 0,\ x^2+y^2+z^2\le a^2,\ \delta(x,y,z)=1$

3.6.9. $S=(x,y,z):0\le x\le 1,\ 0\le y\le 1,\ 0\le z\le 1,\ \delta(x,y,z)=x^2+y^2+z^2$

3.6.10. $S=(x,y,z):0\le x\le 1,\ 0\le y\le 1,\ 0\le z\le 1-x-y,\ \delta(x,y,z)=1$

3.7: Application: Probability and Expected Value

B

3.7.1. Evaluate the integral $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ using anything you have learned so far.

3.7.2. For $\sigma>0$ and $\mu>0$, evaluate $\int_{\infty}^{-\infty}\frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}dx$.

3.7.3. Show that $EY=\frac{n}{n+1}$ in Example 3.18

C

3.7.4. Write a computer program (in the language of your choice) that verifies the results in Example 3.18 for the case $n = 3$ by taking large numbers of samples.

3.7.5. Repeat Exercise 4 for the case when $n = 4$.

3.7.6. For continuous random variables X, Y with joint p.d.f. $f(x, y)$, define the *second moments* $E(X^2)$ and $E(Y^2)$ by

$$E(X^2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 f(x,y)\,dx\,dy \text{ and } E(Y^2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^2 f(x,y)\,dx\,dy, \tag{3.E.3}$$

and the *variances* $\text{Var}(X)$ and $\text{Var}(Y)$ by

$$\text{Var}(X) = E(X^2) - (EX)^2 \text{ and } \text{Var}(Y) = E(Y^2) - (EY)^2. \tag{3.E.4}$$

Find $\text{Var}(X)$ and $\text{Var}(Y)$ for X and Y as in Example 3.18.

3.7.7. Continuing Exercise 6, the correlation ρ between X and Y is defined as

$$\rho = \frac{E(XY) - (EX)(EY)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}, \tag{3.E.5}$$

where $E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\,dx\,dy$. Find ρ for X and Y as in Example 3.18.
(*Note: The quantity* $E(XY) - (EX)(EY)$ *is called the covariance of* X *and* Y.)

3.7.8. In Example 3.17 would the answer change if the interval $(0, 100)$ is used instead of $(0, 1)$? Explain.
